Some Aspects of Fluid Mechanics in Chemical Engineering

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The prediction of the transfers of momentum, energy, and material is of ever-pressing concern to the engineer interested in process design. Recently developed facts concerning the transport of momentum which are of particular use in predicting transfers of energy and material are reviewed, as well as the background of thermodynamics associated with transport processes. The more elemental relationships pertaining to thermal and material transfer in flowing streams are considered, and some of the interrelations of the transport processes are presented.

Processes of interest in chemical engineering include the transfers of momentum, energy, and material. The science of fluid mechanics is primarily concerned with the transfer of momentum in fluid systems. Momentum by nature is a vector quantity; as a result, its proper evaluation involves more complicated mathematical concepts than are encountered in the study of transfers of the scalar quantities, energy and material. In the first third of this century the chemical engineer was concerned chiefly with the application of macroscopic concepts to the prediction of momentum, energy, and material transport in fluid systems. Such interest led to the extensive application of the principles of dimensional analysis (*20*) and to a corollary, widespread use of dimensionless parameters in the correlation of experimental data. These treatments for the most part did not take into account the intensity or scale of turbulence as it influences each of the several transport processes.

With the development of the airplane and consequent rapid advances in the science of fluid mechanics, it was found advantageous to consider momentum transport at a point in the fluid. The work of Horace Lamb and Theodore von Kármán are examples of the early application of the conservation of mass and momentum to the prediction of velocity as a function of the spatial coordinates of fluid systems. Application of such concepts of fluid mechanics, together with the conservations of energy and of matter, to the prediction of energy and material transport in fluid systems was first made by men outside the chemical engineering profession. Most chemical engineers were content until recently to consider all transport phenomena in their macroscopic aspect and treated in a simple manner the variation in molecular properties of a fluid in a conduit with respect to position at a section. These assumptions were usually adequate for the design of equipment used in the accomplishment of physical processes such as fractionation and distillation. Macroscopic methods were applicable also to the design of equipment in which fluids approached physicochemical equilibrium, or in which the approach to chemical equilibrium was sufficiently slow as to be of a time scale different from that associated with the transport processes involved.

Such macroscopic concepts, however, are inadequate for dealing with many processes of widespread and increasing interest, such as jet propulsion and industrial processes in which the exchange between the kinetic energy of a flowing stream and the internal energy of the fluid is large. It has been recognized that in fluids flowing through the nozzles of gas turbines and rocket motors (*26*) deviations from physicochemical equilibrium are often encountered. These deviations are of such a magnitude as to suggest that the process of exchange between internal and kinetic energy within a nozzle or similar device might be used to freeze chemical reactions. Such rapid changes in thermodynamic state have been applied to the manufacture of nitric oxide (*9*) and acetylene (*2, 4*), where the rapid change in internal energy is effected by thermal transfer to solid or liquid phases. It is probable that rapid cooling may be employed to advantage in the manufacture of a variety of other chemical compounds. The prediction of the momentum of reacting fluids as a function of spatial position will, it seems, be of increasing importance to the chemical engineering profession.

This discussion will outline briefly the interrelation of momentum, energy, and material transfer in fluid systems and treat briefly the effect of the level of turbulence upon such transfers. All the following equations are based on the dimensions of force, length, and time, which when taken with temperature serve as the undefined concepts sufficient to describe the phenomena. For convenience, energy is expressed in conventional units, but density has the dimensions pounds seconds squared per foot^4 and specific weight the dimensions pounds per cubic foot. The equations for the sake of simplicity apply to momentum transport in one dimension, unless otherwise noted.

TRANSFER OF MOMENTUM

In a prediction of the transfer of momentum in fluid systems an understanding of the pertinent variables is necessary. Knowledge of these variables, velocity and specific weight, as a function of the spatial coordinates of the system is sufficient to define the local values of the momentum flux. The momentum flux in the x direction can be described by

$$\overset{\circ}{M}_x = \frac{\overset{\circ}{m}_x u_x}{g} = \frac{\sigma u_x^2}{g} \qquad (1)$$

It is essential that it be possible to apply

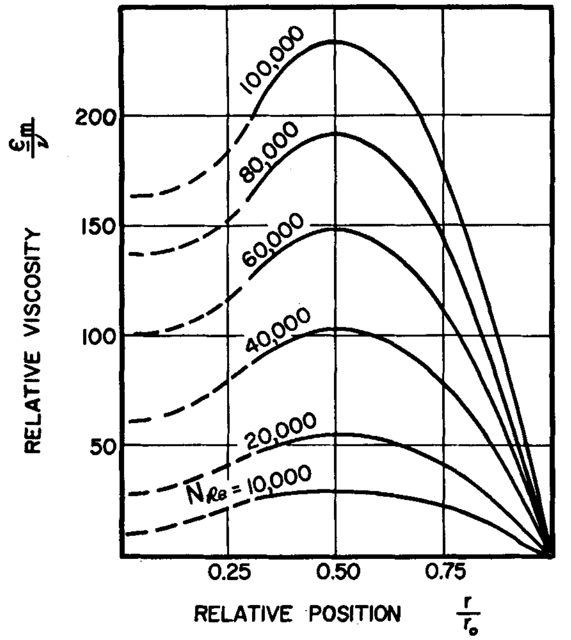

Fig. 1. Variation in eddy viscosity with position and Reynolds number for shear flow in a circular conduit.

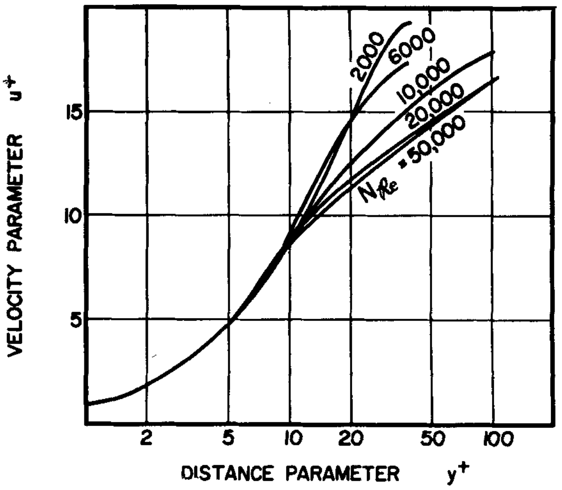

Fig. 2. Effect of distance parameter upon velocity parameter in shear flow.

to the flowing system the equation of continuity, which can be expressed for one-dimensional flow as

$$\left(\frac{\partial\sigma}{\partial\theta}\right)_x = -\left(\frac{\partial\mathring{m}_x}{\partial x}\right) = -\frac{\partial(\sigma u_x)}{\partial x} \qquad (2)$$

The subsonic flow of fluids can be described by two typical kinds of flow, laminar and turbulent. In laminar flow the velocity gradients are related to the viscosity of a Newtonian fluid by

$$\frac{du}{dy} = \frac{\tau_{yx}}{\eta} = \frac{\tau_{yx} g}{\sigma\nu} \qquad (3)$$

In this case the viscosity, which is a molecular property of the fluid, is solely a function of the thermodynamic state of a Newtonian fluid and can be determined from a knowledge of the temperature, pressure, and composition.

Transition to turbulent flow occurs when velocities increase for a given set of physical boundaries and instabilities appear. Turbulent flow can be characterized as a statistical phenomenon in which the situation at any point is random with respect to time. Application of the basic equations of fluid mechanics to turbulent flow is not very useful, except in unusually simple situations. For this reason there has been some interest in a macroscopic measure of the effects of turbulence. This involves the concept of eddy viscosity (*14*).

$$\epsilon_m = \frac{g\tau_{yx}}{\sigma\left(\frac{du_x}{dy}\right)} - \nu \qquad (4)$$

Equation (4) is different from Equation (3) because the total viscosity is the sum of the kinematic viscosity and the eddy viscosity.

$$\underline{\epsilon}_m = \epsilon_m + \nu \qquad (5)$$

The eddy viscosity is a function not only of the state of the fluid but of the condition of flow at and in the vicinity of the point in question. Furthermore the eddy viscosity may not be isotropic.

One of the difficult problems associated with turbulence is the prediction of its rate of growth and decay under particular boundary conditions. This problem adds greatly to the complexity of predicting the flow pattern and the transport characteristics in turbulent flow. In addition the artificially induced turbulence associated with turbulence bars and grids in shear flow materially influences values of the eddy viscosity.

Variations in the relative viscosity $\underline{\epsilon}_m/\nu$ in steady, uniform, turbulent shear flow of fluids in circular conduits have been shown to be a function of the position in the channel and the Reynolds number of the flow (*24*). Such variations are illustrated in Figure 1. From experimental measurements of velocity distributions and pressure gradients values of the relative viscosity as a function of position in flow between parallel plates have been established (*23*). The velocity distribution in the turbulent boundary flow at Reynolds numbers above 20,000 can be conveniently related to the velocity parameter defined by

$$u^+ = \frac{u_x}{\sqrt{\frac{(\tau_{yx})_0 g}{\sigma}}} = K_3 + \frac{1}{K_2}\ln(y^+) = 5.5 + \frac{1}{0.4}\ln(y^+) \qquad (6)$$

The symbol $(\tau_{yx})_0$ refers to the shear at the wall of the circular conduit. It has been found that, contrary to the simple considerations of the principles of similarity (*13*), the velocity parameter u^+ is not a single-valued function of the distance parameter y^+ at transition Reynolds numbers. In this region the relationship varies decidedly with the Reynolds number (Figure 2).

The relationship between the Fanning friction factor and the position in the channel and the total viscosity $\underline{\epsilon}_m$ for Reynolds numbers above 20,000 for flow between parallel plates (*7*) is given by:

$$\frac{\underline{\epsilon}_m}{\nu} = \frac{\epsilon_m + \nu}{\nu} = \frac{l}{l_0}\cosh^2 \cdot\left\{0.0695\,\frac{l_0 - l}{\nu}\,U\sqrt{\frac{f}{2}}\right\} \qquad (7)$$

$$\text{for}\quad y^+ < 27$$

and

$$\frac{\underline{\epsilon}_m}{\nu} = 0.4\,\frac{U}{\nu}\sqrt{\frac{f}{2}}\left(\frac{l}{l_0}\right)(l_0 - l)$$

$$\text{for}\quad y^+ > 27 \quad\text{to}\quad \frac{l}{l_0} < 0.66$$

The first of the expressions is applicable only in the boundary flow, whereas the second may be used throughout all but the central third of the channel.

At the present time it is difficult to predict the influence of turbulence upon nonuniform boundary flow. Application of the Blasius equation (*3*) permits ready evaluation of the velocity as a function of distance from the wall in the case of potential flow in the main stream for a flat plate. Such an approach has been expanded to include the boundary flow for cylinders (*10, 22*) and spheres (*10*). The predicted velocity distributions in the boundary flow about a 2-in. cylinder and a 2-in. sphere for an air stream velocity of 10 ft./sec. are depicted in Figures 3 and 4. The normal distance from the surface to lines of

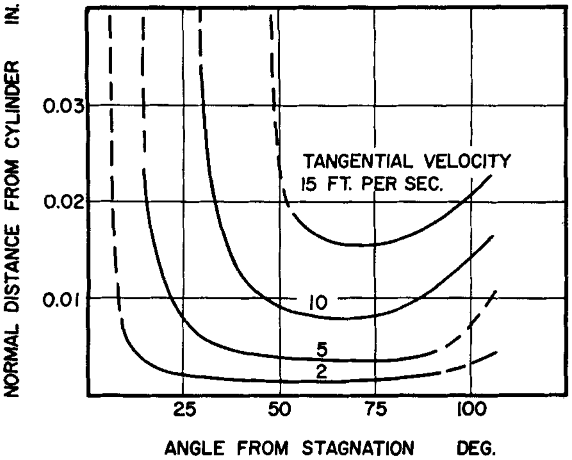

Fig. 3. Estimated velocity distribution in boundary flow about a cylinder.

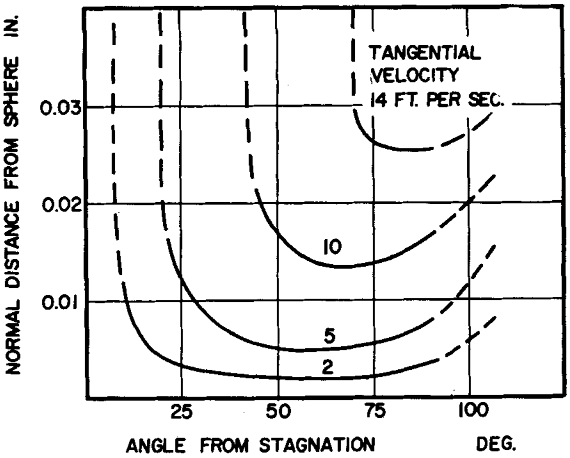

Fig. 4. Estimated velocity distribution in boundary flow about a sphere.

constant velocity is shown in the diagrams as a function of the angle measured from stagnation.

Experimental data illustrating the effect of position upon the velocity in the wake of a cylinder are shown in Figure 5 for a Reynolds number of 4,440 and a gross air velocity of 45 ft./sec. (*1*). The marked complexity in the flow patterns is evident, and furthermore the flow in the wake is distinctly turbulent. Near the point of separation of the boundary flows, as described by Equation (7), there exist regions in which vortex streets are shed from the cylinder. The existence of these vortex streets introduces periodic disturbances in the flow pattern in the wake which are of much lower frequency than the perturbations associated with the turbulent flow. It is obvious that as the scale or intensity of turbulence is increased, numerous inroads into the laminar boundary flows about cylinders and spheres are to be expected. Significant increases in the thermal and material transfer have been found to result from modifications in the level of turbulence (*5, 21*).

THERMODYNAMICS

Many of the concepts of thermodynamics, particularly those relating to the first law, are of importance in the application of fluid mechanics to problems of chemical engineering. The energy transfers to a system may be limited to heat, defined as thermal flux at the boundary, and work, defined as mechanical transfers at the boundary of the system. The defining relations for heat and work are given by

$$\begin{aligned} \underline{q} &= -\int_0^\theta \int_0^A k\left(\frac{\partial T}{\partial z}\right) d\theta \, dA \\ &= m[C_p \, dT + l_p \, dP] \\ &\quad + \sum_1^n \lambda_k \, dm_k - \underline{j} \end{aligned} \tag{8}$$

$$\begin{aligned} \underline{w} &= P \, d\underline{V} - P_a V_a \, dm - \underline{j} \\ &= mP\left[\left(\frac{\partial V}{\partial T}\right)_{P,n} dT \right. \\ &\quad \left. + \left(\frac{\partial V}{\partial P}\right)_{T,n} dP\right] \\ &\quad + \sum_1^n P \bar{V}_k \, dm_k \\ &\quad - P_a V_a \, dm - \underline{j} \end{aligned} \tag{9}$$

The coefficient λ_k is the coefficient of the differential change in weight of component k. It may be evaluated by methods described later in the discussion. The quantity $\underline{j}$ is an infinitesimal that takes into account the mechanical irreversibility associated with changes in total volume of the system. It is considered to be associated with the boundaries. The conservation of energy may be written as

$$\begin{aligned} d\underline{E} &= \underline{q} - \underline{w} + E_a \, dm \\ &= \underline{q} - \underline{w} + \sum_1^n \bar{E}_{ka} \, dm_k \end{aligned} \tag{10}$$

The basic expression for the first law of thermodynamics is

$$\begin{aligned} d\underline{E} &= m\left[\left(\frac{\partial E}{\partial T}\right)_{P,n} dT \right. \\ &\quad \left. + \left(\frac{\partial E}{\partial P}\right)_{T,n} dP\right] + \sum_1^n \bar{E}_k \, dm_k \end{aligned} \tag{11}$$

The foregoing expressions apply to multicomponent systems of variable weight. A combination of Equations (8) to (11) gives a useful expression relating the change in state of a system to the energy transfers across its boundary:

$$\begin{aligned} d\underline{E} &= m\left[\left(\frac{\partial E}{\partial T}\right)_{P,n} dT \right. \\ &\quad \left. + \left(\frac{\partial E}{\partial P}\right)_{T,n} dP\right] \\ &\quad + \sum_1^n \bar{E}_k \, dm_k \\ &= m[C_p \, dT + l_p \, dP] \\ &\quad + \sum_1^n \lambda_k \, dm_k \\ &\quad - mP\left[\left(\frac{\partial V}{\partial T}\right)_{P,n} dT \right. \\ &\quad \left. + \left(\frac{\partial V}{\partial P}\right)_{T,n} dP\right] \\ &\quad - \sum_1^n (P\bar{V}_k - P_a\bar{V}_{ka}) \, dm_k \\ &\quad + \sum_1^n \bar{E}_{ka} \, dm_k \end{aligned} \tag{12}$$

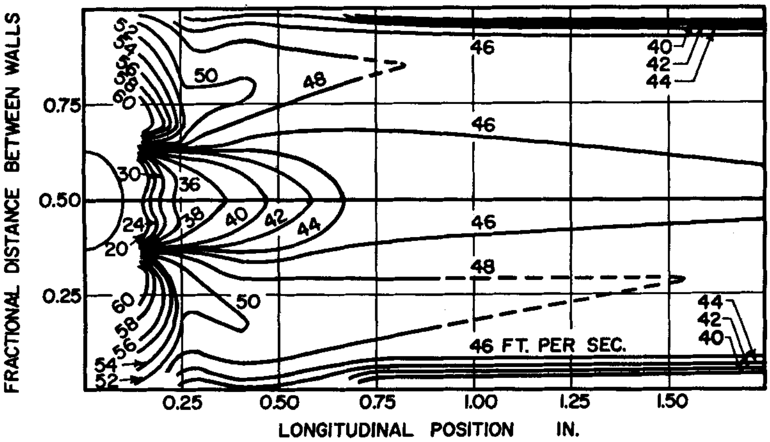

Fig. 5. Variation in velocity throughout the wake of a cylinder.

Equating coefficients one gets

$$m\left(\frac{\partial E}{\partial T}\right)_{P,n} = \left(\frac{\partial \underline{E}}{\partial T}\right)_{P,m} = m\left[C_p - P\left(\frac{\partial V}{\partial T}\right)_{P,n}\right] \tag{13}$$

$$m\left(\frac{\partial E}{\partial P}\right)_{T,n} = \left(\frac{\partial \underline{E}}{\partial P}\right)_{T,m} = m\left[l_p - P\left(\frac{\partial V}{\partial P}\right)_{T,n}\right] \tag{14}$$

and with the defining relationship for the partial enthalpy there results

$$\bar{E}_k - \bar{E}_{ka} = \lambda_k - (P\bar{V}_k - P_a\bar{V}_{ka}) \tag{15}$$

$$\lambda_k = \bar{H}_k - \bar{H}_{ka} \tag{16}$$

$$C_p = \left(\frac{\partial E}{\partial T}\right)_{P,n} + P\left(\frac{\partial V}{\partial T}\right)_{P,n} = \left(\frac{\partial H}{\partial T}\right)_{P,n} \tag{17}$$

$$l_p = \left(\frac{\partial E}{\partial P}\right)_{T,n} + P\left(\frac{\partial V}{\partial P}\right)_{T,n} = -T\left(\frac{\partial V}{\partial T}\right)_{P,n} \tag{18}$$

Equation (16) and the first equalities of Equations (17) and (18) serve as defining relations for the coefficients of change in weight, pressure, and temperature. The second equality of Equation (18) results from application of the test for exactness to relations involving both the first and second laws. Equation (8) serves as a direct and simple means of evaluating the thermal transfers to a system. The corresponding expressions for the second law, when one assumes an identity between the thermodynamic temperature and the absolute temperature, are

$$d\underline{S} = \frac{\underline{q} + \underline{j}}{T} - \frac{1}{T}\sum_{1}^{n} \lambda_k \, dm_k + \sum_{1}^{n} \bar{S}_{ka} \, dm_k \tag{19}$$

$$d\underline{S} = m\left[\left(\frac{\partial S}{\partial T}\right)_{P,n} dT + \left(\frac{\partial S}{\partial P}\right)_{T,n} dP\right] + \sum_{1}^{n} \bar{S}_k \, dm_k \tag{20}$$

The following coefficients may be defined in terms of the second law. The corresponding first-law evaluations are included.

$$C_p = T\left(\frac{\partial S}{\partial T}\right)_{P,n} dT = \left(\frac{\partial H}{\partial T}\right)_{P,n} \tag{21}$$

$$l_p = T\left(\frac{\partial S}{\partial P}\right)_{T,n} = -T\left(\frac{\partial V}{\partial T}\right)_{P,n} \tag{22}$$

$$\bar{S}_k = \left(\frac{\partial \underline{S}}{\partial m_k}\right)_{T,P,m_i} \tag{23}$$

The second equalities in Equations (21) and (22) are derived from simultaneous application of the first and second laws or from the first law alone. There is no expression for the coefficient for the change in entropy with respect to composition. There exists no equivalent to the conservation of energy, and a combination of Equations (19) and (20) does not lead to anything as useful as Equation (12).

From some of the concepts of irreversible thermodynamics it may be readily shown that the existence of shear, transfer of energy, and the transfer of material each results in an entropy source

$$\mathring{\sigma}_{S,V} = \frac{1}{T}\cdot\tau\cdot\left(\frac{\partial u}{\partial y}\right) = \frac{\eta}{T}\left(\frac{\partial u}{\partial y}\right)^2 = J_x X_x \tag{24}$$

$$\mathring{\sigma}_{S,V} = -\frac{\mathring{q}}{T}\left(\frac{\partial \ln T}{\partial x}\right) = k\left(\frac{\partial \ln T}{\partial x}\right)^2 = J_x X_x = \left[k\left(\frac{\partial \ln T}{\partial x}\right)\right]\left[\frac{\partial \ln T}{\partial x}\right] \tag{25}$$

$$\mathring{\sigma}_{S,V} = -\sum_{1}^{n} \frac{\mathring{m}_k}{T}\left(\frac{\partial \mu_k}{\partial x}\right)_{T,P} = \sum_{1}^{n} \frac{D_{\mu,k}}{T}\left(\frac{\partial \mu_k}{\partial x}\right)^2 = \sum_{1}^{n} J_{x,k}X_{x,k} = -\sum_{1}^{n} [\mathring{m}_k]\left[\frac{1}{T}\left(\frac{\partial \mu_x}{\partial x}\right)_{T,P}\right] \tag{26}$$

These expressions are limited to a steady uniform one-dimensional transfer of momentum, energy, and material under nonturbulent conditions. The third equality of the first line of Equation (26), $\sum_1^n (D_{\mu,k}/T)\,(\partial\mu_k/\partial x)^2$, involves a frame of reference moving with a velocity equal to that of the system. It may be readily shown that the velocity of the system does not in itself contribute to the entropy production. The dissipative functions of Equations (24) through (26) indicate some of the ways by which the entropy of a system is increased as a result of momentum, thermal, and material transfers. The Onsager reciprocity relationships afford a simple linear approximation to the interrelation between fluxes, and these relationships can be employed with confidence at states near equilibrium but are not thought to apply with accuracy at steady states remote from equilibrium. Furthermore the concepts of irreversible thermodynamics have not been extended yet to transient processes and therefore are not directly applicable to local turbulent transport or to reaction kinetics in a turbulently flowing medium.

THERMAL TRANSFER

The macroscopic aspects of thermal transfer are well known and are not reviewed in this discussion. The prediction of convective thermal transfer from a macroscopic standpoint has not been so satisfactory as the prediction of the transport of momentum. In the case of molecular transport and in situations involving laminar flow the basic concepts of thermal transfer are well established and can be applied within the limits of

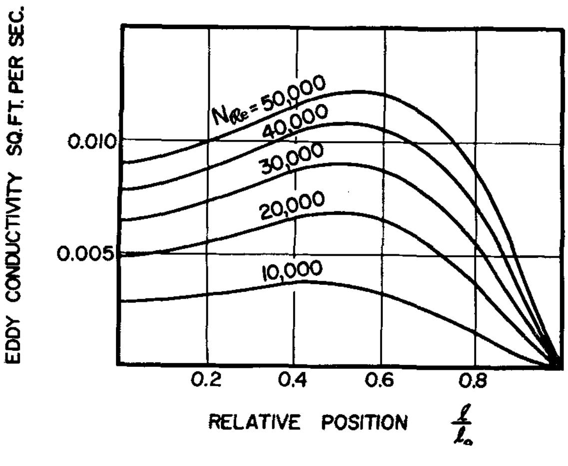

Fig. 6. Variation of eddy conductivity with position and Reynolds number for flow between parallel plates.

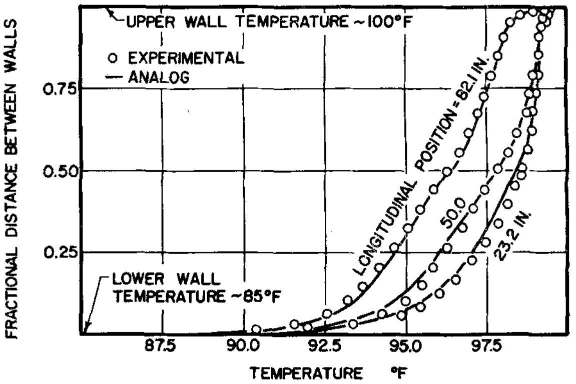

Fig. 7. Temperature distribution in turbulent flow between parallel plates for a gross velocity of 30 ft./sec.

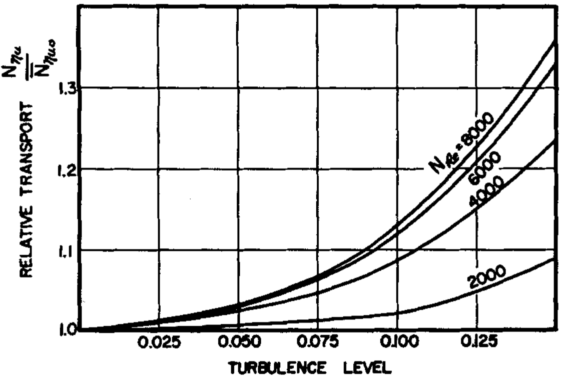

Fig. 8. Effect of turbulence level upon macroscopic thermal transfer.

the solution of the applicable differential equation.

The following equations represent approximations of the actual situation and neglect the differences between the normal-stress tensor and the pressure. For many purposes they prove to be a useful approximation, but care should be exercised in their application to situations where marked divergences between the pressure and the normal-stress tensor exist. In one-dimensional transport with laminar flow the transport of energy at a section may be described by

$$\overset{\circ}{\mathcal{E}} = \overset{\circ}{m}\left(H + h + \frac{u^2}{2g}\right) - k\left(\frac{\partial t}{\partial x}\right) \quad (27)$$

If potential and kinetic energy are neglected, Equation (27) reduces to

$$\overset{\circ}{\mathcal{E}} = \overset{\circ}{m}H - k\left(\frac{\partial t}{\partial x}\right) = \sigma u H - k\,\frac{\partial t}{\partial x} \quad (28)$$

The conservation of energy may be written in the following form if the thermal conductivity is considered a constant:

$$\frac{\partial \sigma \mathcal{E}}{\partial \theta} = -\frac{\partial(\sigma u H)}{\partial x} + k\,\frac{\partial^2 t}{\partial x^2} \quad (29)$$

If thermal transfer by diffusion takes place in two dimensions, Equation (29) may be written for steady, uniform, laminar flow in one dimension as

$$k\left(\frac{\partial^2 t}{\partial y^2}\right) = -k\left(\frac{\partial^2 t}{\partial x^2}\right) + \sigma u_x\,\frac{\partial H}{\partial x} \quad (30)$$

As a matter of interest the exact form of Equation (30) may be shown to be

$$\sigma u_x\,\frac{\partial H}{\partial x} - u_x\,\frac{\partial P}{\partial x} = k\left[\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2}\right] + \Phi \quad (31)$$

It should be noted that Equation (31) contains the dissipation function Φ and the term $-\ u_x\ (\partial P/\partial x)$, which in part compensate each other; thus their omission from Equation (30) does not introduce as large an error as would the omission of only one of the terms. In many situations it is advantageous to consider the internal energy rather than the enthalpy, particularly when marked digression between the normal-stress tensor and the pressure exists.

Equation (30) assumes the following form for cases in which the fluid follows the equation of state of a perfect gas or of a liquid in which the enthalpy is only a function of temperature:

$$k\left(\frac{\partial^2 t}{\partial y^2}\right) = -k\left(\frac{\partial^2 t}{\partial x^2}\right) + \sigma u_x C_p\,\frac{\partial t}{\partial x} \quad (32)$$

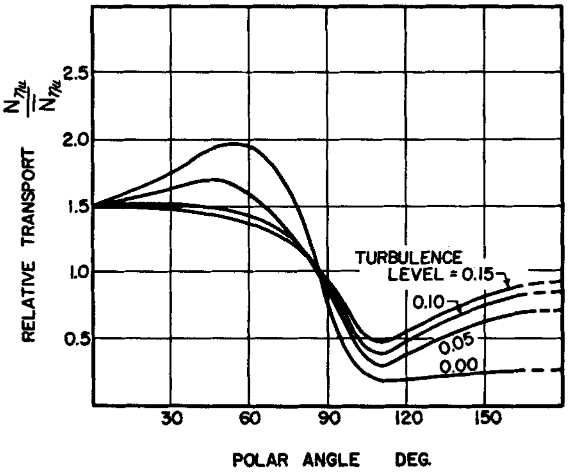

Fig. 9. Effect of turbulence level upon local thermal transfer at an air velocity of 16 ft./sec.

In Equation (32), it is assumed that the transfer of momentum in the system is both steady and uniform.

In the case of turbulent flow use of only the molecular properties of the fluid to describe its local thermal-transport characteristics is not sufficient, and it is necessary to consider eddy conductivity. Figure 6 shows the variation in eddy conductivity with position and Reynolds number during steady uniform flow between parallel plates. The eddy conductivity possesses the same dimensions as the thermometric conductivity and the kinematic viscosity. For turbulent conditions of flow Equation (32) can be rewritten as

$$\frac{\partial}{\partial y}\left[(\epsilon_c)_y\,\frac{\partial t}{\partial y}\right] = -\frac{\partial}{\partial x}\left[(\epsilon_c)_x\left(\frac{\partial t}{\partial x}\right)\right] + u_x\,\frac{\partial t}{\partial x} \quad (33)$$

In Equation (33) it was not assumed that the eddy conductivity was isotropic, and if the Reynolds analogy possesses even qualitative significance, the recent work by Laufer (*16*) would indicate that the eddy conductivity may differ significantly in the longitudinal and normal modes. The importance of the difference between the longitudinal and transverse values of the eddy conductivity is probably of greatest magnitude in the transition region of the boundary flow. In any consideration of the local conditions in turbulent flow the nature of the fluctuations in temperature with time at a given point in the flowing stream becomes important. It is beyond the scope of this paper to discuss the interrelations of the local fluctuations in temperature and velocity. Nevertheless the influence of turbulence upon the transport is in a large measure influenced by the relative magnitudes of these two fluctuations.

Figure 7 illustrates an application of Equation (33) to the prediction of temperature distribution in two-dimensional turbulent flow between parallel plates. In this instance the temperature along one boundary undergoes an abrupt change from 100° to 85°F. and effects a marked change in the temperature distribution within the stream.

If Equation (30) is written for the two-dimensional laminar flow of a reacting system, the following equation is obtained:

$$\frac{\partial \sigma \mathcal{E}}{\partial \theta} = \overset{\circ}{r}\sigma(\Delta E_c) - \frac{\partial(\sigma u_x H)}{\partial x} + k\,\frac{\partial^2 t}{\partial x^2} - \frac{\partial(\sigma u_y H)}{\partial y} + k\,\frac{\partial^2 t}{\partial y^2} \quad (34)$$

In Equation (34) the reaction rate per unit weight of system $\overset{\circ}{r}$ is markedly a function of the state of the system, and its evaluation may not be a simple matter.

Effect of Level of Turbulence

The foregoing discussion is related to thermal transfer in shear flow. Recently the influence of level of turbulence upon thermal and material transport has been investigated, and studies have been made by Sherwood (*19*), Kestin (*15*), Sato (*21*), Short (*25*), and Brown (*5*).

Level of turbulence in the x direction can be conveniently defined as

$$\alpha_r = \frac{\sqrt{\overline{u_{xf}^{\,2}}}}{U} \tag{35}$$

In most situations the turbulence is not isotropic. Figure 8 shows the influence of level of turbulence upon thermal transfer from a 0.5-in. silver sphere. It is apparent that the effect is not nearly so great at the lower Reynolds numbers as at the higher ones. The ordinate in Figure 8 is the ratio of the average Nusselt number for the entire sphere at the indicated level of turbulence and at constant Reynolds number to the same Nusselt number at zero level of turbulence. In some instances there is as much as 40% increase in the thermal transfer with an increase in the level of turbulence to 0.15. The variation in the local transport around a sphere has been studied by Cary (*6*), Lautman and Droege (*17*), and Hsu (*11*). In Figure 9 the ordinate is the ratio of the local Nusselt number to the average Nusselt number for the entire sphere for the conditions of flow in question. The data shown in Figure 9 were obtained with a 0.5-in. silver sphere and at an air velocity of 16 ft./sec. It is apparent that there exists a nearly fivefold variation in the local Nusselt number with variation in the angle from 0 to 180 deg. as measured from stagnation. It is not clear from the experimental data available whether or not the local transport is in fact a continuous function or whether at the point of separation there exists a discontinuous increase in the Nusselt number. The location of the discontinuity, if it exists, is undoubtedly a statistical function of time.

MATERIAL TRANSPORT

In considering material transport it is necessary to take into account the molecular transport resulting from diffusion and the gross transport resulting from the velocity of the phase relative to the frame of reference. The transport of one component by diffusion can be directly related to the concentration gradient of that component and the Fick diffusion coefficient, if the influence of other fields, such as temperature and the gradients of other components, is neglected. To make the presentation relatively straightforward, the influence of other fields is neglected in the following discussion. As a matter of interest the variation in the Fick diffusion coefficient of methane with the molecular weight of various paraffinic stagnant components is shown in Figure 10. A nearly thirtyfold variation in the Fick diffusion coefficient occurs in passing from a stagnant component, such as propane, to a relatively heavy oil.

Laminar Flow

The material transport of component k can be established from

$$\mathring{m}_{k,x} = \sigma_k u - D_{Fk}\frac{\partial \sigma_k}{\partial x} \tag{36}$$

From the conservation of material as applied to component k there follows, when D_{Fk} is assumed to be a constant,

$$\frac{\partial \sigma_k}{\partial \theta} = -\frac{\partial \mathring{m}_k}{\partial x} = -\frac{\partial \sigma_k u}{\partial x} + D_{Fk}\frac{\partial^2 \sigma_k}{\partial x^2} \tag{37}$$

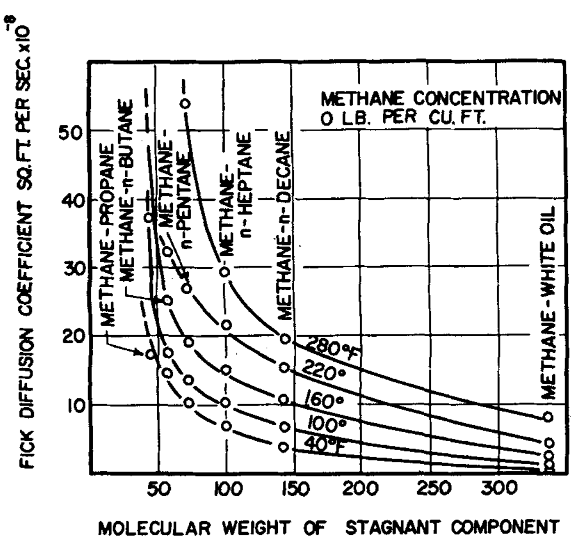

Fig. 10. Variation in Fick diffusion coefficient for methane with molecular weight of the paraffinic stagnant component.

Equations (36) and (37) are restricted to the one-dimensional transport of component k. In the case of the steady state transport of component k in two dimensions it follows that

$$\frac{\partial \mathring{m}_{kx}}{\partial x} = -\frac{\partial \mathring{m}_{ky}}{\partial y} \tag{38}$$

Combining Equations (37) and (38), neglecting all gross motions in the y direction other than those introduced by diffusion, and remembering that

$$\sum_1^n D_{Fk}\left(\frac{\partial \sigma_k}{\partial x}\right) = 0,$$

one gets

$$\mathring{m}_{ky} = -\left(1 + \frac{\sigma_k}{\sigma_i}\right) D_{Fk}\left(\frac{\partial \sigma_k}{\partial y}\right) - \frac{\sigma_k}{\sigma_i} D_{Fi}\frac{\partial \sigma_i}{\partial x} - \frac{\sigma_k}{\sigma_i} D_{Fk}\frac{\partial \sigma_k}{\partial x} \tag{39}$$

The last two terms in Equation (39) arise from consideration of two-dimensional diffusion. When one neglects those two terms, a combination of Equation (39) with Equations (36) and (38) results for steady conditions in

$$\frac{\partial \sigma_k u}{\partial x} - D_{Fk}\left(\frac{\partial^2 \sigma_k}{\partial x^2}\right) = D_{Fk}\frac{\partial}{\partial y}\left[\left(1 + \frac{\sigma_k}{\sigma_i}\right)\left(\frac{\partial \sigma_k}{\partial y}\right)\right] \tag{40}$$

Equation (40) may be solved by a number of the standard methods for solution of partial differential equations, for example that proposed by Longwell (*18*). Some simplifications are usually required.

Turbulent Flow

In the case of turbulent flow the total diffusivity is substituted for the Fick diffusion coefficient, and Equation (40) becomes

$$\frac{\partial \sigma_k u}{\partial x} - \frac{\partial}{\partial x}\left[(\epsilon_{dk})_x \frac{\partial \sigma_k}{\partial x}\right] = \frac{\partial}{\partial y}\left[(\epsilon_{dk})_y\left(1 + \frac{\sigma_k}{\sigma_i}\right)\left(\frac{\partial \sigma_k}{\partial y}\right)\right] \tag{41}$$

In this instance, since the total eddy diffusivity is distinctly a function of

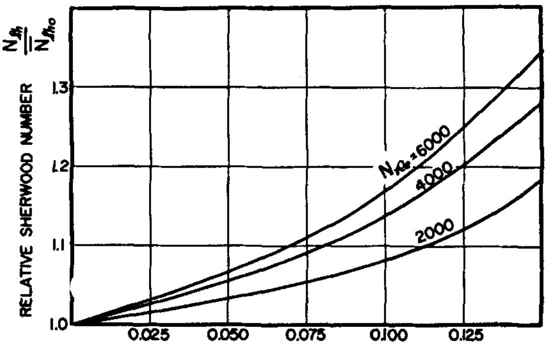

Fig. 11. Effect of level of turbulence upon macroscopic material transfer.

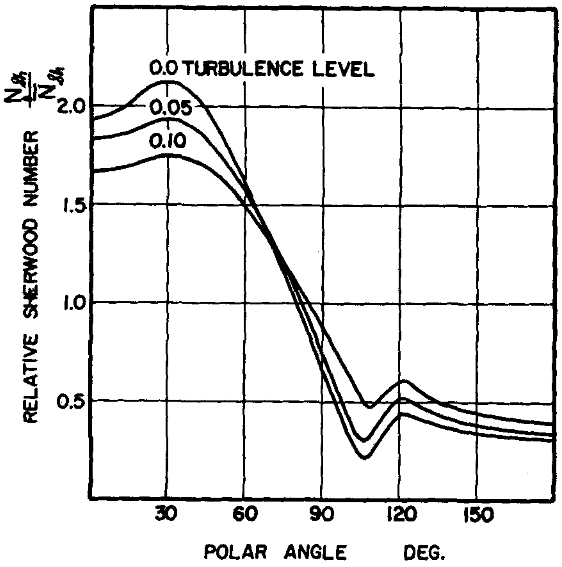

Fig. 12. Effect of level of turbulence upon local material transfer at an air velocity of 16 ft./sec.

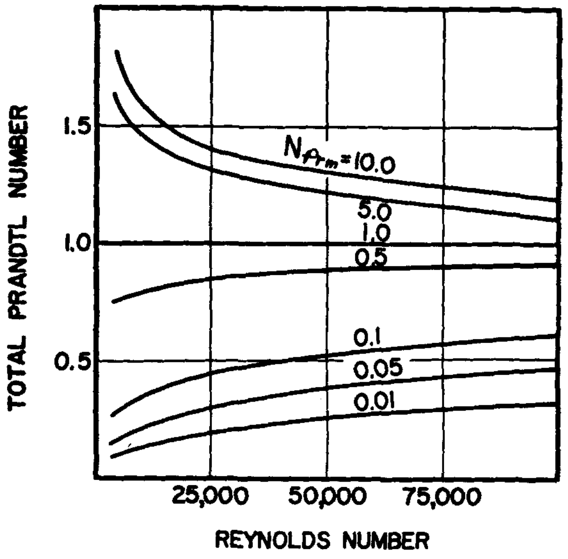

Fig. 13. Effect of Reynolds number upon total Prandtl number.

position in the flowing stream and may not be isotropic, it has been kept within the partial differential rather than being removed from it, as was done in the case of the Fick diffusion coefficient in Equation (40). It is apparent that if the variation in velocity with respect to the x direction is neglected, Equation (41) is analogous to Equation (33).

Effect of Level of Turbulence

From the earlier consideration of the influence of level of turbulence as defined by Equation (35), material transport would be expected to be markedly influenced by the level of turbulence. Figure 11 illustrates the variation in the average Sherwood number with level of turbulence for several different Reynolds numbers. Again it is seen that there is approximately a 35 % increase in the material transfer with an increase in the level of turbulence to about 0.15; again the effect is much more pronounced at the higher Reynolds numbers. The data for Figure 11 are based upon evaporation from 0.5-in. porous spheres (*5*).

The corresponding local material transport from spheres as determined from the local thermal transfer to the surface is presented in Figure 12. There exists nearly a fivefold variation in the local transport with polar angle measured from the stagnation point. The influence of the level of turbulence upon the distribution of material transport is significant. Again it is possible that there is a discontinuous change in the rate of material transfer at the point of separation, and the polar angle corresponding to this separation probably varies statistically with respect to time. The values shown in Figure 12 do not take into account the thermal conduction within the sphere which may have influenced the results slightly. It should be recognized that there exist significant local fluctuations in the concentration of the diffusing component in turbulent flow. The interrelation of these fluctuations with the local fluctuations in velocity is a complicated matter and is not fully understood at present. It is beyond the scope of this paper to consider the nature of these fluctuations, although they undoubtedly control the effectiveness of turbulent transport.

Analogies between Momentum, Thermal, and Material Transport

The conventional Reynolds analogy assumes that transport in turbulent flow is identical for momentum, internal energy, and material. From this assumption it follows that

$$\epsilon_m = \epsilon_c = \epsilon_d \tag{42}$$

In this situation the total viscosity, conductivity, and diffusivity are related to the kinematic viscosity, thermometric conductivity, and Fick diffusion coefficient by

$$\underline{\epsilon}_m = \epsilon_m + \nu \tag{43}$$

$$\underline{\epsilon}_c = \epsilon_c + \mathrm{K} \tag{44}$$

$$(\underline{\epsilon}_d)_k = (\epsilon_d)_k + D_{Fk} \tag{45}$$

Recently Corcoran and Sage (*8*), utilizing some of the ideas of Jenkins (*12*), showed that there is a relatively small variation in the total Prandtl number with Reynolds number. With this in view it might be possible to describe another type of limiting analogy:

$$\begin{aligned} \phi\left(N_{Re}, \frac{r}{r_0}\right) &\approx \frac{(\underline{\epsilon}_m)_y}{\nu} = \frac{(\epsilon_m)_y + \nu}{\nu} \\ &\approx \frac{(\underline{\epsilon}_c)_y}{\mathrm{K}} = \frac{(\epsilon_c)_y + \mathrm{K}}{\mathrm{K}} \\ &\approx \Big|_1^n \frac{(\underline{\epsilon}_d)_k}{D_{Fk}} = \Big|_1^n \frac{(\epsilon_d)_k + D_{Fk}}{D_{Fk}} \end{aligned} \tag{46}$$

The function of Reynolds number and position in the stream is the same as that shown in Figure 1. Equation (46) is equivalent to the assumption that the turbulent Prandtl and Schmidt numbers are identical with the molecular Prandtl and Schmidt numbers and that in the end all transport is dependent upon molecular processes. It is well known that the Reynolds analogy is not true, and it is equally probable that the analogy set forth by Equation (46) is not true. At relatively low Reynolds numbers, however, and for fluids which possess Prandtl numbers that are not far different from unity, it is probable that Equation (46), which is the equivalent of

$$\underline{N}_{Pr} = N_{Pr} \tag{47}$$

$$\underline{N}_{Sc} = N_{Sc} \tag{48}$$

is as descriptive of the actual situation as the conventional Reynolds analogy. On the basis of Jenkin's theory, modified by recent experimental data, the effect of Reynolds number upon the total Prandtl number is indicated in Figure 13. As a first order of approximation it may be assumed that the same values apply to the total Schmidt number.

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NOTATION

A = area, sq. ft.
C_p = isobaric heat capacity, B.t.u./(lb.)(°R.)
cosh = hyperbolic cosine
D_F = Fick diffusion coefficient, sq. ft./sec.
D_μ = diffusion coefficient based on chemical potentials
E = internal energy, B.t.u./lb.
$\bar{E}$ = partial internal energy, B.t.u./lb.
$\underline{E}$ = total internal energy, B.t.u.
$\mathcal{E}$ = kinetic and internal energy, B.t.u./lb.
$\mathring{\mathcal{E}}$ = energy flux, B.t.u./(sq. ft.)(sec.)
f = Fanning friction factor
g = acceleration due to gravity, ft./sec.^{2}
H = enthalpy, B.t.u./lb.
$\bar{H}$ = partial enthalpy, B.t.u./lb.
h = height, ft.
J = a flux
$\underline{j}$ = total friction, B.t.u.
K_2, K_3 = constants
k = thermal conductivity, B.t.u./(sec.) (ft.) (°F.)
l = distance from center of channel, ft. or in.
l_0 = distance from center to wall, ft. or in.
ln = natural logarithm
l_p = latent heat of pressure change, (B.t.u./lb.)/(lb./sq. in.)
$\mathring{M}$ = momentum flux, lb./sq. ft.
m = weight of material, lb.
$\mathring{m}$ = weight rate of flow, lb./(sq. ft.) (sec.)
N_{Nu} = Nusselt number
$\bar{N}_{Nu}$ = space-average Nusselt number
$\bar{N}_{Nu0}$ = space-average Nusselt number at zero turbulence level
N_{Pr} = Prandtl number
$\underline{N}_{Pr}$ = total Prandtl number
N_{Prm} = molecular Prandtl number
N_{Re} = Reynolds number
N_{Sc} = Schmidt number
$\underline{N}_{Sc}$ = total Schmidt number
N_{Sh} = Sherwood number
$\bar{N}_{Sh}$ = space-average Sherwood number
$\bar{N}_{Sh0}$ = space-average Sherwood number at zero turbulence level
n = nth value or index
n = weight fraction
P = pressure, lb./sq. in.
$\underline{q}$ = total heat, B.t.u.
$\mathring{q}$ = local thermal flux, B.t.u./(sq. ft.) (sec.)
$\mathring{r}$ = reaction rate, sec.$^{-1}$
r = radial distance, ft. or in.
r_0 = radius of conduit, ft. or in.
S = entropy, B.t.u./(lb.) (°R.)
$\bar{S}$ = partial entropy, B.t.u./(lb.) (°R.)
$\underline{S}$ = total entropy, B.t.u./°R.
T = absolute temperature, °R.
t = temperature, °F.
U = gross velocity, ft./sec.
u = point velocity, ft./sec.
u^+ = velocity parameter,

$$u^+ = u/\sqrt{g(\tau_{yx})_0/\sigma}$$

$\bar{u}_{xf}$ = mean fluctuating velocity in the x direction, ft./sec.
V = volume, cu. ft./lb.
$\bar{V}$ = partial volume, cu.ft./lb.
$\underline{V}$ = total volume, cu. ft.
$\underline{w}$ = total work, B.t.u.
X = a potential
x = downstream distance along axis of flow, ft. or in.
y = distance normal to axis of flow, ft. or in.
y_d = vertical distance from wall, ft. or in.
y^+ = distance parameter,

$$y^+ = (y_d/\nu)\sqrt{(g/\sigma)(\tau_{yx})_0}$$

z = distance normal to boundary

Greek Letters

α_τ = turbulence level
Δ = difference
ΔE_c = change in internal energy resulting from a chemical reaction, B.t.u./lb.
ϵ_c = eddy conductivity, sq. ft./sec.
$\underline{\epsilon}_c$ = total conductivity, sq. ft./sec.
ϵ_d = eddy diffusivity, sq. ft./sec.
$\underline{\epsilon}_d$ = total diffusivity, sq. ft./sec.
ϵ_m = eddy viscosity, sq. ft./sec.
$\underline{\epsilon}_m$ = total viscosity, sq. ft./sec.
η = absolute viscosity, (lb.)/(sec.)/sq. ft.
θ = time, sec.
K = thermometric conductivity, sq. ft./sec.
λ = defined in Equation (16)
μ = chemical potential, B.t.u./lb.
ν = kinematic viscosity, sq. ft./sec.
σ = specific weight, lb./cu. ft.
σ = concentration, lb./cu. ft.
$\mathring{\sigma}$ = rate of entropy production
τ = shear, lb./sq. ft.
Φ = dissipation function
$\phi(\ )$ = function of

Subscripts

a = material added
i = ith property or index
j = component j
k = component k
m = weight
n = weight fraction
o = reference
P = pressure
S = entropy
T = temperature
V = volume
x = in x direction
y = in y direction

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